ORTHOGONAL-BASED SIXTH ORDER ALGORITHM AS INITIAL VALUE SOLVER

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Abstract: For many years, the choice of basis function to develop an algorithm has been to choose from the existing polynomials. In this work, a zero-stable continuous hybrid scheme which exactly integrates second order initial value problems in ordinary differential equations is constructed. A new class of polynomials with recursive formula is employed as trial function. Findings from the analysis of the scheme show that it is accurate, efficient and convergent as its solutions accurately produce analytical solutions.

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Introduction

Polynomials appear in a wide variety of areas of mathematics and science. For example, they are used to form equations, which encode a wide range of problems, from elementary word problems to complicated problems.In science, they are used to define polynomial functions, which appear in settings ranging from basic chemistry and physics to economics and social science; they are used in calculus and numerical analysis to approximate other functions. Notable among the well-known polynomials are the orthogonal polynomials. Orthogonal polynomial sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product. The first orthogonal polynomials were the Legendre polynomials. Then came the Chebyshev polynomials, the general Jacobi polynomials, the Hermite and the Laguerre polynomials. All these classical orthogonal polynomials play an important role in many applied problems.

Asymptotic formulae for orthogonal polynomials were first discovered by G. Szegö, Szego (1975). Lanczos, C. (1938) introduced Chebyshev polynomials as trial function.Several researchers have employed these polynomials as trial functions to formulate algorithms (see Shampine and Watts (1969), Tanner (1979), Dahlquist (1979), Jator (2007), Awoyemi (1991)). We note that the zeros of the Chebyshev's polynomial of the first kind, the zeros of the Legendre's polynomial and even other polynomials can also be chosen but our keen interest is to derive a new class of polynomials for general use. The desire for increase in accuracy and efficiency of numerical methods has motivated several authors to propose methods for solving initial value problems (Ramos (2016), Vigo-Aguiar and Ramos (2006), Simos (2002)).

In this work, we shall employ a non-negative weight function to construct a class of orthogonal polynomials which will serve as trial functions to formulate numerical algorithms for the solution of second order initial value problems.

Construction of orthogonal basis function

Let the function $q_n(x)$, the quantity to be evaluated be defined as;

$$q_r(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \qquad (1)$$

on the real interval [a, b], **where** $q_r(x)$ must satisfy the orthogonal property

$$< q_m(x), q_n(x) > = \int_a^b w(x) q_m(x) q_n(x) dx = 0, \; m \neq n \qquad (2)$$

For the purpose of constructing the basis function, we adopt the approach discussed extensively in Adeyefa and Adeniyi (2015) and use additional property (the normalization) $q_n(1)=1$ where our weight functions is defined as $w(x) = x^2 - 1$.

For r = 0 in (1), $q_0(x) = C_0^{(0)}$

Using the normalization equation, $q_0(1) = C_0^{(0)} = 1$

Hence, $q_0(x) = 1$

For r = 1 in (1),

$$q_1(x) = C_0^{(1)} + C_1^{(1)} x \qquad (3)$$

Applying the normalization equation, (3) gives;

$$C_0^{(1)} + C_1^{(1)} = 1 \qquad (4)$$

and $< q_0, q_1 > = \int_{-1}^{1} w(x) q_0(x) q_1(x) dx$

which implies;

$$\frac{4}{3} C_0^{(1)} = 0 \qquad (5)$$

Solving (4) and (5) and substituting the outcomes into (3), we have;

$$q_1(x) = x \qquad (6)$$

When r = 2 in (5),

$$q_2(x) = C_0^{(2)} + C_1^{(2)} x + C_2^{(2)} x^2 \qquad (7)$$

By normalization definition, (7) gives;

$$C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1 \qquad (8)$$

Considering $< q_0, q_2 > = \int_{-1}^{1} (x+1) q_0(x) q_2(x) dx$

$= 0,$

we have;

$$\frac{4}{3} C_0^{(2)} + \frac{4}{15} C_2^{(2)} = 0 \qquad (9)$$

Taking;

$$< q_1, q_2 > = \int_{-1}^{1} (x+1) q_1(x) q_2(x) dx$$
$$= 0$$

we obtain;

$$\frac{4}{15} C_1^{(2)} = 0 \qquad (10)$$

Solving (8), (9), (10) and substituting the resulting values into (7), we have;

$$q_2(x) = \frac{1}{4}(5x^2 - 1) \qquad (11)$$

We obtain, for r = 3 in (1), $q_3(x) = \frac{1}{4}(7x^3 - x)$.

In the same vein, $q_n(x), n \geq 4$ are developed. The first eleven of this class of orthogonal polynomials are listed hereunder.

$$\left.\begin{aligned}
q_0(x) &= 1 \\
q_1(x) &= x \\
q_2(x) &= \frac{1}{4}(5x^2 - 1) \\
q_3(x) &= \frac{1}{4}(7x^3 - x) \\
q_4(x) &= \frac{1}{8}(21x^4 - 14x^2 + 1) \\
q_5(x) &= \frac{1}{8}(33x^5 - 30x^3 + 5x) \\
q_6(x) &= \frac{1}{64}(429x^6 - 495x^4 + 135x^2 - 5) \\
q_7(x) &= \frac{1}{64}(715x^7 - 1001x^5 + 385x^3 - 35x) \\
q_8(x) &= \frac{1}{128}(2431x^8 - 4004x^6 + 2002x^4 - 308x^2 + 7) \\
q_9(x) &= \frac{1}{128}(4199x^9 - 7956x^7 + 4914x^5 - 1092x^3 + 63x) \\
q_{10}(x) &= \frac{1}{512}(29393x^{10} - 62985x^8 + 46410x^6 - 13650x^4 + 1365x^2 - 21)
\end{aligned}\right\} \qquad (12)$$

This set of polynomials shall be referred to as ADEM-B Orthogonal polynomials.

In the spirit of Golub and Fischer (1992), equation (12) must satisfy three-term recurrence relation

$$c_j p(t) = (t - a_j) p_{j-1}(t) - b_j p_{j-2}(t),\ j = 1,2,...,\ p_{-1}(t) = 0,\ p_0(t) \equiv p_0$$

where b_j, $c_j > 0$ for $j \geq 1$ (b_1 is arbitrary).

$$c_j p(t) = (n+3)P_{n+1}(x),\ (t - a_j) p_{j-1}(t) = (2n+3)xP_n(x),\ b_j p_{j-2}(t) = nP_{n-1}(x), n = 1,2,...$$

Therecursion formula for these orthogonal polynomials is therefore given as

$$P_{n+1}(x) = \frac{1}{n+3}\left[(2n+3)xP_n(x) - nP_{n-1}(x)\right],\ n \geq 1,\ P_0(x) = 1,\ P_1(x) = x$$

This relation, along with the two polynomials $P_0(x)$ and $P_1(x)$, allows the new set of polynomials to be generated recursively.

In the next section, we shall develop an algorithm to integrate second order differential equations where polynomials $q_n(x)$ shall be employed as basis function. Thereafter, the analysis of the method for convergence and implementation of the method through some test problems shall be presented. Finally, conclusion shall be made.

Formulation of the Method

In this section, our aim is to derive a continuous scheme from which a set of block formula is developed. To make this happen, we shall seek an approximant

$$y(x) = \sum_{r=0}^{s+k-1} a_r q_r(x) \qquad (13)$$

to obtain the solution of second order initial value problems in ordinary differential equations. Transforming $q_n(x)$ in interval [-1,1] to [0,1], we have $x = \frac{2X - 2x_n - ph}{ph}$, where p varies as the method to be developed.

Here, we formulate a step method and in (13), s and k are points of interpolation and collocation respectively. The procedure involves interpolating (13) at points s = $0, \frac{1}{3}$ and collocating the second derivative of (13) at points k = $0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, 1$. The a_r, r = 0(1)6 from the resulting system of equations are obtained as;

$$
\left.
\begin{aligned}
a_0 &= h^2\left(\frac{1027}{27216}f_n-\frac{137}{63}f_{n+\frac{1}{6}}+\frac{934375}{217728}f_{n+\frac{1}{5}}-\frac{68512}{25515}f_{n+\frac{1}{4}}+\frac{4009}{6720}f_{n+\frac{1}{3}}+\frac{421}{653184}f_{n+1}-\frac{1}{2}y_n+\frac{3}{2}y_{n+\frac{1}{3}}\right)\\
a_1 &= h^2\left(\frac{26329}{213840}f_n-\frac{122077}{17325}f_{n+\frac{1}{6}}+\frac{33160625}{2395008}f_{n+\frac{1}{5}}-\frac{2417696}{280665}f_{n+\frac{1}{4}}+\frac{27349}{14784}f_{n+\frac{1}{3}}+\frac{58613}{25660800}f_{n+1}-\frac{3}{2}y_n+\frac{3}{2}y_{n+\frac{1}{3}}\right)\\
a_2 &= h^2\left(\frac{31}{180}f_n-\frac{2736}{275}f_{n+\frac{1}{6}}+\frac{30625}{1584}f_{n+\frac{1}{5}}-\frac{5888}{495}f_{n+\frac{1}{4}}+\frac{1071}{440}f_{n+\frac{1}{3}}+\frac{149}{39600}f_{n+1}\right)\\
a_3 &= h^2\left(\frac{1619}{12012}f_n-\frac{1080}{143}f_{n+\frac{1}{6}}+\frac{696875}{48048}f_{n+\frac{1}{5}}-\frac{1024}{117}f_{n+\frac{1}{4}}+\frac{1337}{8008}f_{n+\frac{1}{3}}+\frac{557}{144144}f_{n+1}\right)\\
a_4 &= h^2\left(\frac{667}{13104}f_n-\frac{1179}{275}f_{n+\frac{1}{6}}+\frac{509375}{104832}f_{n+\frac{1}{5}}-\frac{2272}{819}f_{n+\frac{1}{4}}+\frac{2637}{5824}f_{n+\frac{1}{3}}+\frac{1403}{524160}f_{n+1}\right)\\
a_5 &= h^2\left(-\frac{17}{2640}f_n+\frac{27}{55}f_{n+\frac{1}{6}}+\frac{4375}{4224}f_{n+\frac{1}{5}}+\frac{32}{45}f_{n+\frac{1}{4}}-\frac{567}{3520}f_{n+\frac{1}{3}}+\frac{79}{63360}f_{n+1}\right)\\
a_6 &= h^2\left(-\frac{1}{65}f_n+\frac{3096}{3575}f_{n+\frac{1}{6}}-\frac{1875}{1144}f_{n+\frac{1}{5}}+\frac{2048}{2145}f_{n+\frac{1}{4}}-\frac{477}{2860}f_{n+\frac{1}{3}}+\frac{31}{85800}f_{n+1}\right)\\
a_7 &= h^2\left(-\frac{6}{1001}f_n+\frac{1296}{5005}f_{n+\frac{1}{6}}-\frac{1875}{4004}f_{n+\frac{1}{5}}+\frac{256}{1001}f_{n+\frac{1}{4}}-\frac{81}{2002}f_{n+\frac{1}{3}}+\frac{1}{20020}f_{n+1}\right)
\end{aligned}
\right\}\quad(14)
$$

Substituting (14) into (13) yields the continuous implicit method;

$$y(x)=\alpha_0(x)y_n+\alpha_{\frac{1}{3}}(x)y_{n+\frac{1}{3}}+h^2(\beta_k(x)f_{n+k}),\ k=0,\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},1 \qquad (15)$$

Evaluating equation (15) at x = x_{n+m}, m=$\frac{1}{6},\frac{1}{5},\frac{1}{4},1$ yields the discrete equations;

$$\left.\begin{aligned}
y_{n+1} &= \frac{1193}{2430}h^2 f_n - \frac{6232}{225}h^2 f_{n+\frac{1}{6}} + \frac{104375}{1944}h^2 f_{n+\frac{1}{5}} - \frac{119552}{3645}h^2 f_{n+\frac{1}{4}} + \frac{199}{30}h^2 f_{n+\frac{1}{3}} + \frac{2171}{145800}h^2 f_{n+1} - 2y_n + 3y_{n+\frac{1}{3}} \\
y_{n+\frac{1}{6}} &= -\frac{113}{155520}h^2 f_n - \frac{37}{1350}h^2 f_{n+\frac{1}{6}} + \frac{6875}{248832}h^2 f_{n+\frac{1}{5}} - \frac{52}{3645}h^2 f_{n+\frac{1}{4}} + \frac{61}{69120}h^2 f_{n+\frac{1}{3}} \\
&\quad - \frac{7}{18662400}h^2 f_{n+1} + \frac{1}{2}y_n + \frac{1}{2}y_{n+\frac{1}{3}} \\
y_{n+\frac{1}{5}} &= -\frac{22013}{37968750}h^2 f_n - \frac{77768}{3515625}h^2 f_{n+\frac{1}{6}} + \frac{209}{9720}h^2 f_{n+\frac{1}{5}} - \frac{725248}{56953125}h^2 f_{n+\frac{1}{4}} + \frac{281}{468750}h^2 f_{n+\frac{1}{3}} \\
&\quad - \frac{671}{2278125000}h^2 f_{n+1} + \frac{2}{5}y_n + \frac{3}{5}y_{n+\frac{1}{3}} \\
y_{n+\frac{1}{4}} &= -\frac{1787}{4976640}h^2 f_n - \frac{407}{28800}h^2 f_{n+\frac{1}{6}} + \frac{111875}{7962624}h^2 f_{n+\frac{1}{5}} - \frac{1183}{116640}h^2 f_{n+\frac{1}{4}} + \frac{41}{245760}h^2 f_{n+\frac{1}{3}} \\
&\quad - \frac{103}{597196800}h^2 f_{n+1} + \frac{1}{4}y_n + \frac{3}{4}y_{n+\frac{1}{3}}
\end{aligned}\right\} \quad (16)$$

To develop the block method from the continuous scheme, we adopt the general block formula proposed in Shampine and Watts (1969) in the normalized form given as

$$A^{(0)}Y_m = ey_m + h^{\mu-\lambda}df(y_m) + h^{\mu-\lambda}bF(y_m) \quad (17)$$

Evaluating the first derivative of (15) at $x = x_{n+j}$, $j = 0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, 1$, substituting the resulting equations and equation (16) into (17) and solving simultaneously gives a block formulae represented as

$$\left.\begin{aligned}
y_{n+\frac{1}{6}} &= \frac{18107}{3265920}h^2 f_n + \frac{199}{3150}h^2 f_{n+\frac{1}{6}} - \frac{445625}{5225472}h^2 f_{n+\frac{1}{5}} + \frac{524}{15309}h^2 f_{n+\frac{1}{4}} - \frac{1829}{483840}h^2 f_{n+\frac{1}{3}} \\
&\quad + \frac{541}{391910400}h^2 f_{n+1} + y_n + \frac{1}{6}hy'_n \\
y_{n+\frac{1}{5}} &= \frac{22789}{3281250}h^2 f_n + \frac{236736}{2734375}h^2 f_{n+\frac{1}{6}} - \frac{383}{3360}h^2 f_{n+\frac{1}{5}} + \frac{223744}{4921875}h^2 f_{n+\frac{1}{4}} - \frac{43713}{8750000}h^2 f_{n+\frac{1}{3}} \\
&\quad + \frac{1427}{787500000}h^2 f_{n+1} + y_n + \frac{1}{5}hy'_n \\
y_{n+\frac{1}{4}} &= \frac{1297}{143360}h^2 f_n + \frac{2727}{22400}h^2 f_{n+\frac{1}{6}} - \frac{35625}{229376}h^2 f_{n+\frac{1}{5}} + \frac{631}{10080}h^2 f_{n+\frac{1}{4}} - \frac{783}{114688}h^2 f_{n+\frac{1}{3}} \\
&\quad + \frac{127}{51609600}h^2 f_{n+1} + y_n + \frac{1}{4}hy'_n \\
y_{n+\frac{1}{3}} &= \frac{64}{5103}h^2 f_n + \frac{856}{4725}h^2 f_{n+\frac{1}{6}} - \frac{36875}{163296}h^2 f_{n+\frac{1}{5}} + \frac{7424}{76545}h^2 f_{n+\frac{1}{4}} - \frac{47}{5040}h^2 f_{n+\frac{1}{3}} \\
&\quad + \frac{43}{12247200}h^2 f_{n+1} + y_n + \frac{1}{3}hy'_n \\
y_{n+1} &= \frac{37}{70}h^2 f_n - \frac{4752}{175}h^2 f_{n+\frac{1}{6}} + \frac{11875}{224}h^2 f_{n+\frac{1}{5}} - \frac{2048}{63}h^2 f_{n+\frac{1}{4}} + \frac{3699}{560}h^2 f_{n+\frac{1}{3}} + \frac{751}{50400}h^2 f_{n+1} + y_n + hy'_n \\
y''_{n+\frac{1}{6}} &= \frac{3269}{77760}hf_n + \frac{623}{900}hf_{n+\frac{1}{6}} - \frac{108125}{124416}hf_{n+\frac{1}{5}} + \frac{1232}{3645}hf_{n+\frac{1}{4}} - \frac{421}{11520}hf_{n+\frac{1}{3}} + \frac{121}{9331200}hf_{n+1} + y''_n \\
y''_{n+\frac{1}{5}} &= \frac{7879}{187500}hf_n + \frac{55188}{78125}hf_{n+\frac{1}{6}} - \frac{407}{480}hf_{n+\frac{1}{5}} + \frac{5248}{15625}hf_{n+\frac{1}{4}} - \frac{9099}{250000}hf_{n+\frac{1}{3}} + \frac{97}{7500000}hf_{n+1} + y''_n \\
y''_{n+\frac{1}{4}} &= \frac{323}{7680}hf_n + \frac{2241}{3200}hf_{n+\frac{1}{6}} - \frac{625}{768}hf_{n+\frac{1}{5}} + \frac{43}{120}hf_{n+\frac{1}{4}} - \frac{189}{5120}hf_{n+\frac{1}{3}} + \frac{1}{76800}hf_{n+1} + y''_n \\
y''_{n+\frac{1}{3}} &= \frac{203}{4860}hf_n + \frac{164}{225}hf_{n+\frac{1}{6}} - \frac{6875}{7776}hf_{n+\frac{1}{5}} + \frac{1664}{3645}hf_{n+\frac{1}{4}} - \frac{7}{720}hf_{n+\frac{1}{3}} + \frac{7}{583200}hf_{n+1} + y''_n \\
y''_{n+1} &= \frac{139}{60}hf_n - \frac{3132}{25}hf_{n+\frac{1}{6}} + \frac{23125}{96}hf_{n+\frac{1}{5}} - \frac{2176}{15}hf_{n+\frac{1}{4}} + \frac{2241}{80}hf_{n+\frac{1}{3}} + \frac{317}{2400}hf_{n+1} + y''_n
\end{aligned}\right\} \quad (18)$$

Equation (18) is our desired block method of which its basic properties shall be discussed briefly. The scheme cannot solve IVPs of order n (n > 2) except it is reduced to system of order two equations.

Analysis of the Method

Order and error constant

Following Henrici (1962), the approach adopted in Fatunla (1991, 1994) and Lambert (1973),we define the local truncation error associated with equation (18) by the difference operator

$$L[y(x):h] = \sum_{j=0}^{k}\left[\alpha_j y(x_n + jh) - h^3\beta_j f(x_n + jh)\right] \qquad (19)$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding (19) in Taylor series about the point x, we obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + \ldots + C_{p+3}h^{p+3}y^{p+3}(x)$$

where the C_0 , C_1 , C_2 , $C_p, \ldots$ are obtained as

$$C_0 = \sum_{j=0}^{k}\alpha_j, \qquad C_1 = \sum_{j=1}^{k} j\alpha_j, \qquad C_2 = \frac{1}{2!}\sum_{j=1}^{k} j^2\alpha_j,$$

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q\alpha_j - q(q-1)(q-2)\sum_{j=1}^{k}\beta_j j^{q-3}\right]$$

In the spirit of Lambert (1973), equations (16) and (18) are of order p if

$C_0 = C_1 = C_2 = \ldots C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$

The $C_{p+2} \neq 0$ is called the error constant and $C_{p+2}h^{p+2}y^{p+2}(x_n)$ is the principal local truncation error at the point x_n.

According to the definition above, equations (16) and (18) are all of order 6 with the error constants

$$C_{P+2} = \left[\frac{-17921}{1322697600} \; \frac{821}{5079158784000} \; \frac{26357}{20667150000000} \; \frac{32909}{433421549568000} \right]^T$$

and

$$C_{P+2} = \begin{bmatrix} \frac{-37}{27216000} & \frac{3083}{5079158784000} & \frac{-169}{212625000000} & \frac{-1921}{1783627776000} & \frac{-61}{39680928000} \\ \frac{-199}{35271936000} & \frac{-383}{68040000000} & \frac{-631}{111476736000} & \frac{-47}{8817984000} & \frac{-751}{108864000} \end{bmatrix}^T$$

respectively.

Zero stability of the method

According to Lambert (1973), a linear multistep method is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the method, we present (18) in vector notation form of column vectors $e = (e_1 \ldots e_r)^T$, $d = (d_1 \ldots d_r)^T$, $y_m = (y_{n+1} \cdots y_{n+r})^T$, $F(y_m) = (f_{n+1} \cdots f_{n+r})^T$ and matrices $A = (a_{ij})$, $B = (b_{ij})$.

Thus, equation (18) forms the block formula

$$A^0 y_m = hBF(y_m) + A^1 y_n + hdf_n \qquad (20)$$

Where h is a fixed mesh size within a block.

Hence, based on the definition above, the scheme is zero stable.

Region of absolute stability of the main methods

For the region of absolute stability, the following definitions are considered.

Given the stability polynomial

$$\pi(z, \bar{h}) = \rho(z) - \bar{h}\sigma(z) = 0 \qquad (21)$$

where $\bar{h} = h^2\lambda^2$ and $\lambda = \dfrac{df}{dy}$ are assumed constants.

The method (16) is said to be absolutely stable if for a given $\bar{h}$ all the roots z_s of (21) satisfy $|z_s| < 1$, s=1,2,…n, where $\bar{h} = \lambda h$

Definition: The region $\Re$ of the complex $\bar{h}$-plane such that the roots of $\pi(z, \bar{h}) = 0$ lies within the unit circle whenever $\bar{h}$ lies in the interior of the region is called the region of absolute stability.

Remark: Let $\Re$ be the boundary of the region $\Re$. Since the roots of the stability polynomial are continuous functions of $\bar{h}$, $\bar{h}$ will lie on $\Re$ when one of the roots of the $\pi(z, \bar{h}) = 0$ lies on the boundary of the unit circle. Thus we define (21) in terms of Euler's number, $\exp i\theta$,as follows;

$$\pi(\exp(i\theta), h) = \rho(\exp(i\theta) - \bar{h}\sigma(\exp(i\theta)) = 0 \qquad (22)$$

So that, the locus of the boundary $\Re$ is given by

$$\bar{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \qquad (23)$$

From (16), for y_{n+1}, the first and second characteristic polynomials are as follows

$$\rho(z) = z - 3z^{\frac{1}{3}} + 2 \qquad (24)$$

and,

$$\sigma(z) = \frac{2171}{145800} z + \frac{199}{30} z^{\frac{1}{3}} - \frac{119552}{3645} z^{\frac{1}{4}} + \frac{104375}{1944} z^{\frac{1}{5}} - \frac{6232}{225} z^{\frac{1}{6}} + \frac{1193}{2430} \qquad (25)$$

so that the boundary of the region of absolute stability is

$$\bar{h}(\theta) = \frac{\rho(z)}{\sigma(z)} = \frac{145800(z - 3z^{\frac{1}{3}} + 2)}{2171z + 967140z^{\frac{1}{3}} - 4782080z^{\frac{1}{4}} + 7828125z^{\frac{1}{5}} - 4038336z^{\frac{1}{6}} - 71580} \qquad (26)$$

where $z = e^{i\theta}$.

Thus, the region of absolute stability is shown as Fig. 1.

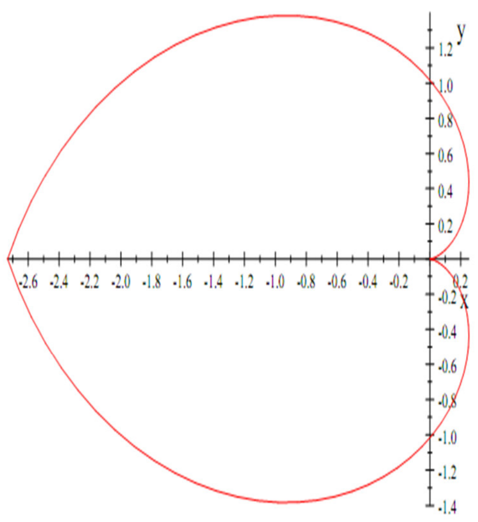

Fig. 1: Region of absolute stability of the method

Consistency of the method

According to Lambert (1973), a linear multistep method is said to be consistent if it has order at least one. Owing to this definition, equations (16) and (18) are consistent.

Convergency of the method

According to the theorem of Dahlquist (1979), the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence it is convergent.

Numerical Experiment

Problem 1:

$$y'' - x(y')^2 = 0, \; y(0) = 1, \; y'(0) = \frac{1}{2}, \; h = 0.0025$$

$$Exact\ Solution: y(x) = 1 + \frac{1}{2}\ln(\frac{2+x}{2-x})$$

Problem 2:

$$y'' = y', \; y(0) = 0, \; y'(0) = -1, \; h = 0.1$$

$$Exact\ Solution: y(x) = 1 - \exp(x)$$

Problem 3:

$$y'' = y + x\exp(3x), \; y(0) = -\frac{3}{32}, \; y'(0) = -\frac{5}{32}, \; h = 0.0025$$

$$Exact\ Solution: y(x) = \frac{4x - 3}{32\exp(-3x)}$$

Table 1: Comparison of exact and approximate solution of Problem 1

X	Analytical Solution	Approximate Solution	Error
0.0025	1.00125000065104227700	1.00125000065104227700	0
0.005	1.00250000520835286470	1.00250000520835286470	0
0.0075	1.00375001757827331690	1.00375001757827331690	0
0.01	1.00500004166729167780	1.00500004166729167780	0
0.0125	1.00625008138211573520	1.00625008138211573520	0
0.015	1.00750014062974628450	1.00750014062974628450	0
0.0175	1.00875022331755040640	1.00875022331755040650	0
0.02	1.01000033335333476200	1.01000033335333476210	1.0e-19
0.0225	1.01125047464541890790	1.01125047464541890800	1.0e-19
0.025	1.01250065110270863570	1.01250065110270863580	1.0e-19

Table 2: Comparison of exact and approximate solution of Problem 2

X	Analytical Solution	Approximate Solution	Error
0.1	-0.10517091807564762480	-0.10517091807566185734	1.423254e-14
0.2	-0.22140275816016983390	-0.22140275816027624933	1.0641543e-13
0.3	-0.34985880757600310400	-0.34985880757629589142	2.9278742e-13
0.4	-0.49182469764127031780	-0.49182469764186245872	5.9214092e-13
0.5	-0.64872127070012814680	-0.64872127070115432592	1.02617912e-12
0.6	-0.82211880039050897490	-0.82211880039212889554	1.61992064e-12
0.7	-1.01375270747047652160	-1.01375270747287867870	2.4021571e-12
0.8	-1.22554092849246760460	-1.22554092849587357400	3.4059694e-12
0.9	-1.45960311115694966380	-1.45960311116161897590	4.6693121e-12
1.0	-1.71828182845904523540	-1.71828182846528090740	6.235672e-12

Table 3: Comparison of exact and approximate solution of Problem 3

X	Analytical Solution	Approximate Solution	Error
0.0025	-0.09414091576184863991	-0.09414091576184863991	2e-21
0.005	-0.09453240414233882994	-0.09453240414233882994	4e-21
0.0075	-0.09492445160838763703	-0.09492445160838763702	5e-21
0.01	-0.09531704439070030914	-0.09531704439070030914	4e-21
0.0125	-0.09571016848098074811	-0.09571016848098074811	1e-21
0.015	-0.09610380962911336911	-0.09610380962911336911	2e-21
0.0175	-0.09649795334031607435	-0.09649795334031607435	0
0.02	-0.09689258487226406553	-0.09689258487226406553	2e-21
0.0225	-0.09728768923218421752	-0.09728768923218421752	4e-21
0.025	-0.09768325117391973292	-0.09768325117391973293	8e-21

Tables 1, 2 and 3 display the accuracy of the numerical results for initial value problems. Figs. 1, 2 and 3 reveal the desirability of the method since on the graph, no deviation is noticed at point.

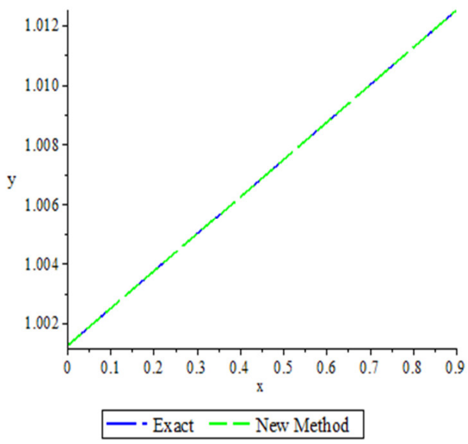

Fig. 2: Graphical comparison of the analytical solution and the solution of the new method using Problem 1.

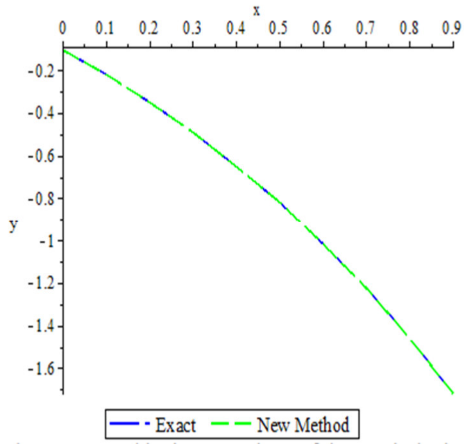

Fig. 3: Graphical comparison of the analytical solution and the solution of the new method using Problem 2.

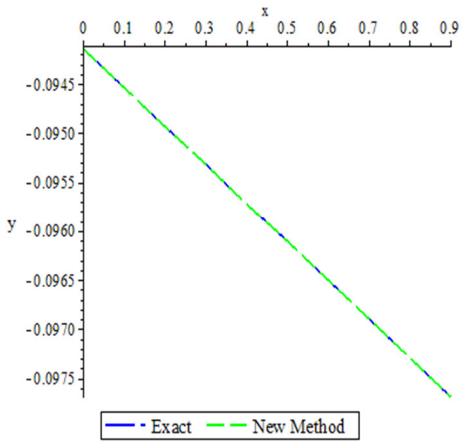

Fig. 4: Graphical comparison of the analytical solution and the solution of the new method using Problem 3.

Conclusion

We have developed a technique to construct orthogonal polynomials using weight function, $w(x) = x^2 - 1$. Formulation of numerical integrators using the generated polynomials has been demonstrated. It has been shown that the introduced construction method can also be straightforwardly applied to obtain both implicit and explicit schemes. No comparison with existing methods is made as it is obvious that the method produces the analytical solution. We therefore recommend the technique for polynomial construction and general use of the polynomials as we hope to extend the approach to solve boundary value problems.

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